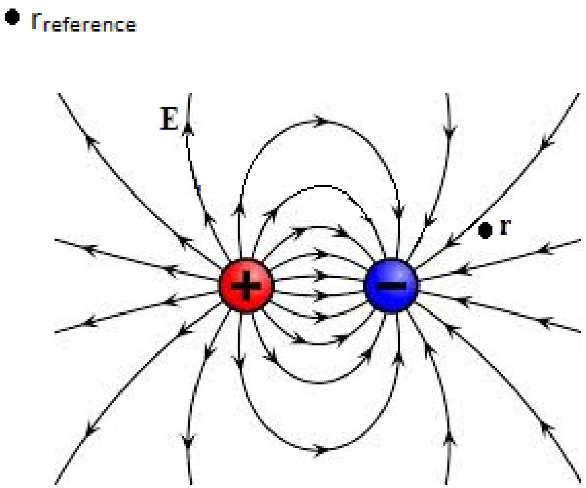
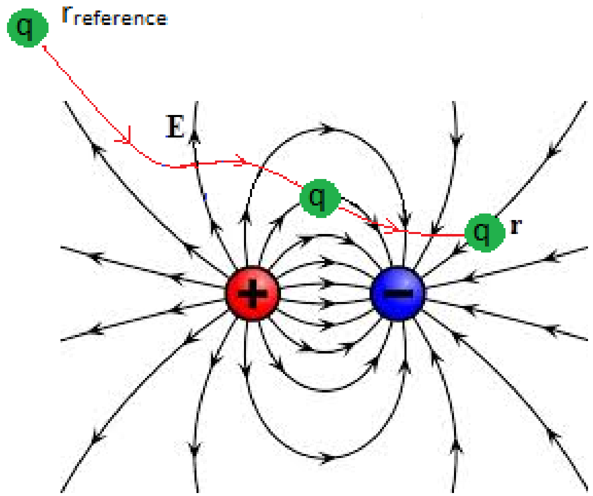
A.5 Electric Potential Energy

Getting closer to applying this stuff…the last topic comprising part of the theoretical foundation of electric fields is the concept of electric potential energy. It is ultimately this energy which we wish to harness and use to make all sorts of things: from lightbulbs to generators and motors.

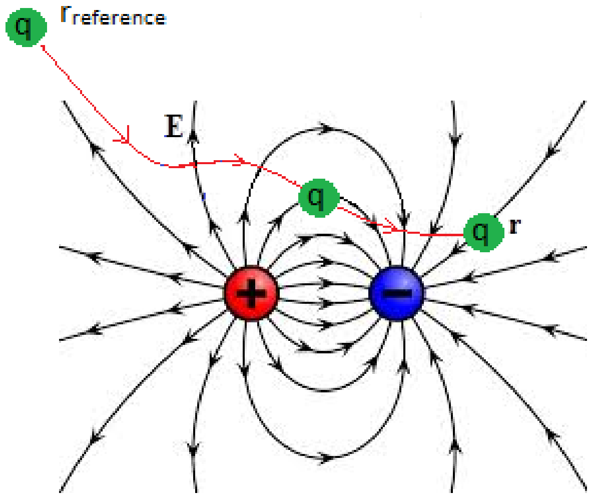
So we begin with this. Suppose we have an electric field.



And suppose we have a (green) charge which we move through this field from a reference point to another point.



Now when you do (+)work on something you impart energy to it, and when you do (–)work on something, it imparts energy to you.



Same holds true for the electric field, the energy imparted to it (by you ultimately) is the (–)work it does on the charge.

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So we make this definition: ***the electric potential energy, PE_E , a charge q stores in an electric field, E, is the (-) work E does on q as it moves from the reference point $r_{reference}$ (usually taken to be infinitely far away) to the point r.***

This is no different than how PE_g was defined in PHY 141:

$$PE_g = -W_{gravity}$$
$$= -\int_{r_{reference}}^{r} \mathbf{F}_g \cdot d\mathbf{r}$$
$$= -\int_0^y (-mg)dy$$
$$= mg\int_0^y dy$$
$$= mgy$$

Or how PE_{spring} was defined:

$$PE_{spring} = -W_{spring}$$
$$= -\int_{r_{reference}}^{r} \mathbf{F}_{spring} \cdot d\mathbf{r}$$
$$= -\int_0^x (-kx)\,dx$$
$$= k\int_0^x xdx$$
$$= \frac{1}{2}kx^2$$

Or how this is defined. There is no one-size-fits-all formula for PE_E since the electric fields we can create, and forces they exert are multifarious. Still, we can relate it quite simply to something we already know how to calculate:

$$PE_E = -W_{electric}$$
$$= -\int_{r_{reference}}^{r} \mathbf{F}_E \cdot d\mathbf{r}$$
$$= -\int_{r_{reference}}^{r} q\mathbf{E} \cdot d\mathbf{r}$$
$$= q\left(-\int_{r_{reference}}^{r} \mathbf{E} \cdot d\mathbf{r}\right)$$
$$= qV$$

So here we go:

$$PE_E = qV$$

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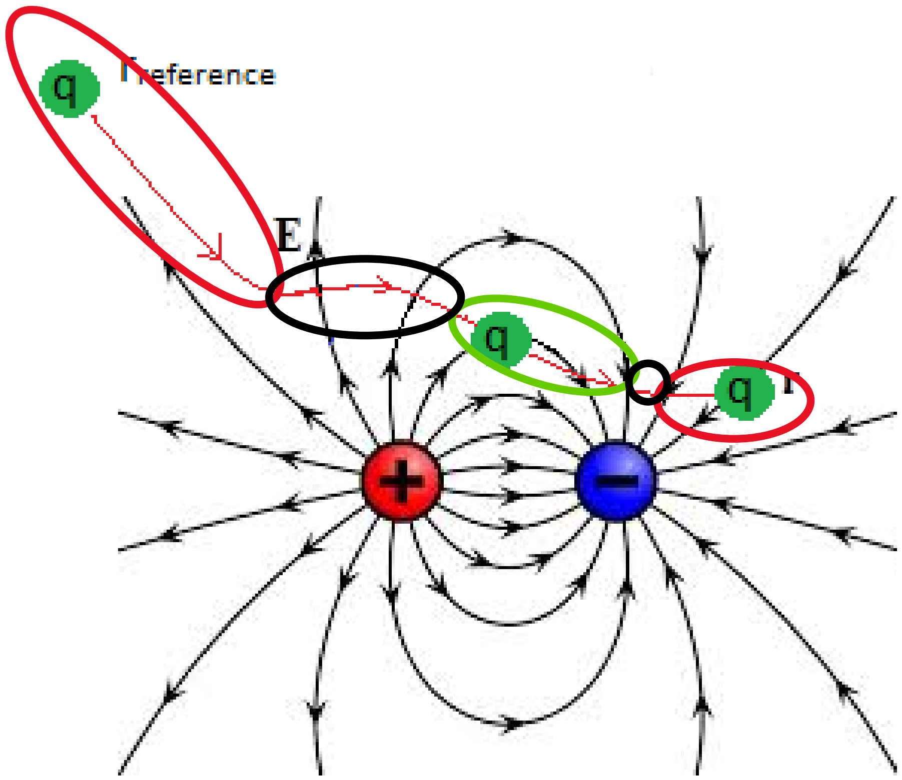

Suppose our our green charge is positive….

(a) Where is it putting energy into the E field?

where it is being decelerated, i.e., going against F_E.

(b) Where is it taking energy out of the field?

where it is being accelerated, i.e., going with F_E

(c) Where is it doing neither?

where it is going at constant speed, i.e., going perpendicular to F_E

Suppose our our green charge is negative, how do things change?

red and green areas switch, black stays same.

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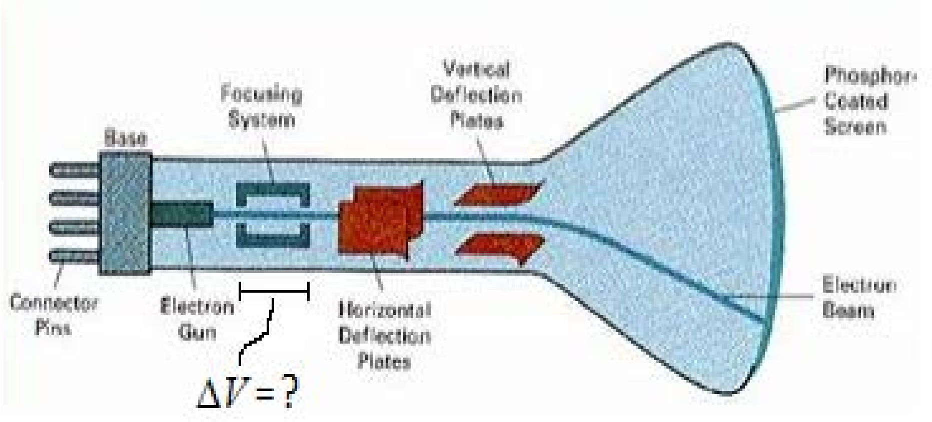

Consider the cathode-ray tube here, which is/was used to project images on oscilloscopes, and the only kind of TV I can afford. On the left, the *electron gun* basically consists of a hot metallic filament off of which electrons pop, much like how water evaporates from a puddle. The electrons that happen to be (or are *made to be* with magnetic fields) going more or less straight will make it into the *focusing system* compartment which consists of a negative and positive charged plate on the left and right (with holes in them to allow the electrons to enter and leave). These plates set up an electric field E (with associated potential difference ΔV) which dramatically accelerates the electrons forward. This electron beam will then be forced up/down and right/left as desired by the electric fields generated by the *deflection plates.* Finally the beam will hit the desired spot on the phosphor coated screen and cause a chemical transition which will release a photon, which you'll see as an image.

Blah blah blah. Anyway, all I want to know is this. What potential difference between the plates is necessary to accelerate the electrons from basically rest, to 9×10^{6} m/s? Remember the mass of an electron is $m = 9.11\times10^{-31}$kg.

So we use the work energy equation, taking our initial point to be when the electrons enter the focusing system on the left, and our final point to be when they leave it on the right:

$$\sum W_{n.c.} = \Delta KE + \Delta PE_E$$

$$0 = \left[\frac{1}{2}mv^2 - \frac{1}{2}m(0)^2\right] + q\Delta V$$

$$q\Delta V = -\frac{1}{2}mv^2$$

$$\Delta V = -\frac{1}{2}\frac{mv^2}{q}$$

$$= -\frac{1}{2}\frac{(9.11\times10^{-31})(9\times10^{6})^2}{-1.6\times10^{-19}} = 231\text{V}$$

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To initiate the radioactive decay of a Li isotope to another element we need to bombard it with protons. The radius of the nucleus is roughly 2.3fm.

(a) If the potential difference between the plates is 500V, how close will it get to the Li nucleus?

$$\sum W_{n.c.} = \Delta PE_E$$

$$0 = q\Delta V + q\Delta V_{Li}$$

$$0 = (e)(-500) + (e)\left[\frac{kq_{Li}}{r_f} - \frac{kq_{Li}}{r_i}\right]$$

$$0 = (e)(-500) + (e)\left[\frac{k(3e)}{r} - \cancel{\frac{k(3e)}{a\,lot}}\right]$$

$$500 = \frac{3ke}{r} \longrightarrow r = \frac{3ke}{500} = \frac{3(9\times10^{9})(1.6\times10^{-19})}{500} = 8.6\times10^{-12}\,\text{m}$$

(b) To initiate radioactive decay, the proton needs to get inside the nucleus. What potential difference would be required for this?

Same thing…

$$\sum W_{n.c.} = \Delta PE_E$$

$$0 = q\Delta V + q\Delta V_{Li}$$

$$0 = e\Delta V + e\left(\frac{k(3e)}{2.3\,\text{fm}} - \cancel{\frac{k(3e)}{a\,lot}}\right)$$

$$\Delta V = -\frac{k(3e)}{2.3\times10^{-15}} = -1.9\times10^{6}\,\text{V}$$

(c) Would answer change if we changed distance between the plates? No

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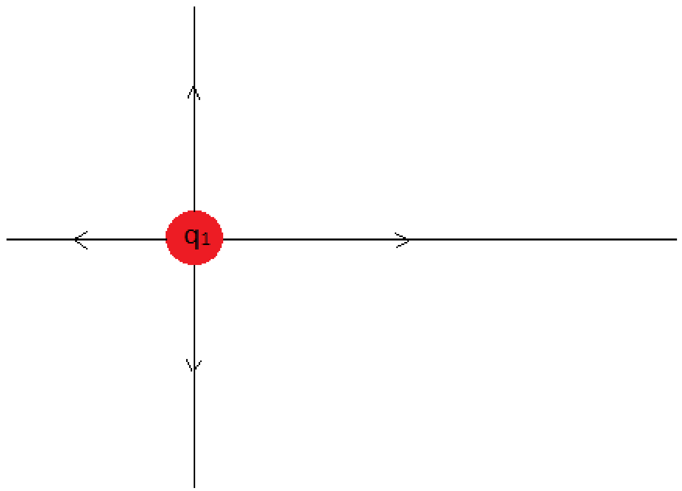
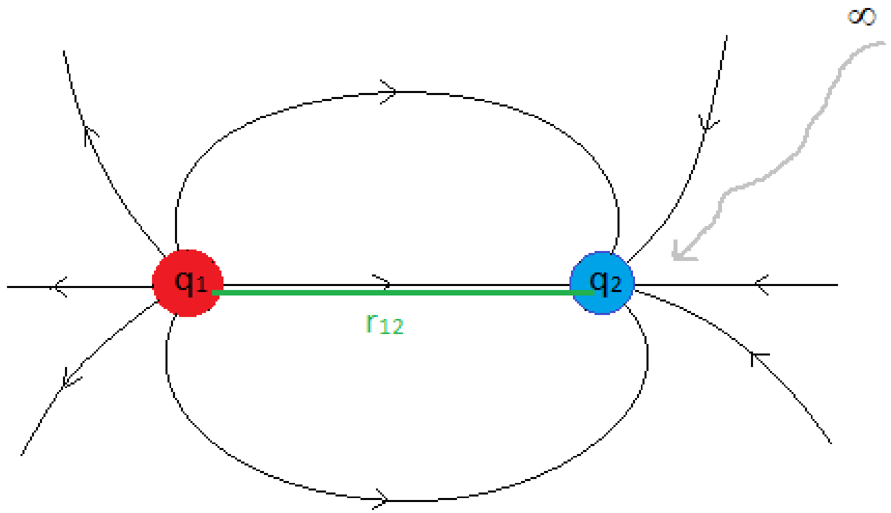
As a charge moves around in an electric field it puts energy in and/or takes energy out. So the electric field is a repository of energy; it's teeming with energy one might say, were one to wax lyrical. A natural question to ask is, 'what's the total energy in the electric field'? So glad you asked.

An electric field is created in a region by bring charges together into that region. So we basically have to figure out how much work it takes to bring the charges together: this would constitute the energy of the field itself. So let's bring the charges in, one by one, and see how much work is required.





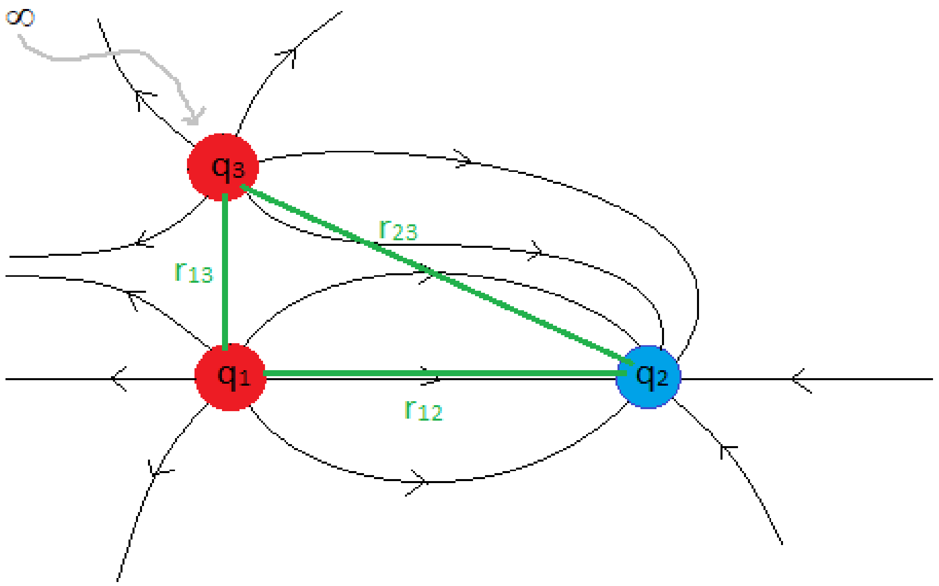
Work required to bring first charge in from the reference point (which we'll take to be infinity) is zero, as there is no field yet to fight against.

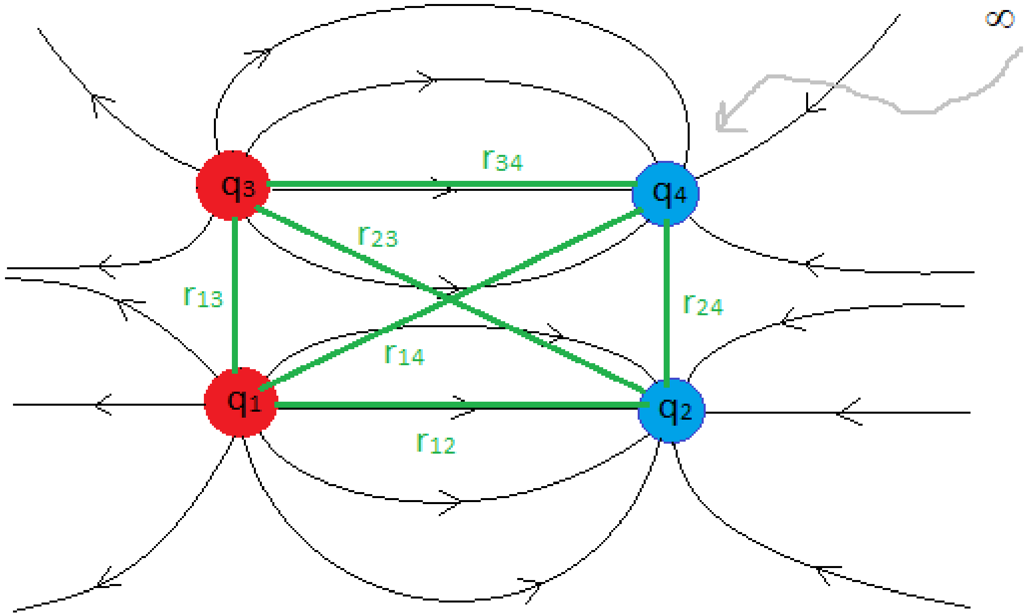
Work required to bring second charge in from reference point (infinity, in case you forgot) is:

$$W_2 = \Delta PE_E = q_2 \Delta V_1$$

$$= q_2\left(\frac{kq_1}{r_{12}} - \frac{kq_1}{\infty}\right) = \frac{kq_1 q_2}{r_{12}}$$

A.5 Electric Potential Energy





Work required to bring third charge in from infinity is:

$$W_3 = q_3 \Delta V_{12}$$
$$= q_3 \left[\Delta V_1 + \Delta V_2 \right]$$
$$= q_3 \left[\left(\frac{kq_1}{r_{13}} - \frac{kq_1}{\infty} \right) + \left(\frac{kq_2}{r_{23}} - \frac{kq_2}{\infty} \right) \right]$$
$$= q_3 \left[\frac{kq_1}{r_{13}} + \frac{kq_2}{r_{23}} \right] = \frac{kq_1q_3}{r_{13}} + \frac{kq_2q_3}{r_{23}}$$

Work required to bring fourth charge in from infinity is:

$$W_4 = q_4 \Delta V_{123}$$
$$= q_4 \left[\Delta V_1 + \Delta V_2 + \Delta V_3 \right]$$
$$= q_4 \left[\left(\frac{kq_1}{r_{14}} - \frac{kq_1}{\infty} \right) + \left(\frac{kq_2}{r_{24}} - \frac{kq_2}{\infty} \right) + \left(\frac{kq_3}{r_{34}} - \frac{kq_3}{\infty} \right) \right]$$
$$= q_4 \left[\frac{kq_1}{r_{14}} + \frac{kq_2}{r_{24}} + \frac{kq_3}{r_{34}} \right] = \frac{kq_1q_4}{r_{14}} + \frac{kq_2q_4}{r_{24}} + \frac{kq_3q_4}{r_{34}}$$

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We'll stop here as we have done enough work (like my pun?) to establish the pattern. So the work required to assemble these four charges is:

$$W = \begin{matrix} W_1 \\ + \\ W_2 \\ + \\ W_3 \\ + \\ W_4 \end{matrix} = \begin{matrix} 0 \\ + \\ \dfrac{kq_1q_2}{r_{12}} \\ + \\ \dfrac{kq_1q_3}{r_{13}} + \dfrac{kq_2q_3}{r_{23}} \\ + \\ \dfrac{kq_1q_4}{r_{14}} + \dfrac{kq_2q_4}{r_{24}} + \dfrac{kq_3q_4}{r_{34}} \end{matrix}$$

In general, if we have N particles, we could say this:

$$PE_E = \sum_{i<j}^{N} \frac{kq_iq_j}{r_{ij}} \qquad \text{Energy stored in the electric field}$$

From this formula we can derive a slightly cooler equivalent expression that explicitly relates PE_E to the electric field itself. But the derivation would require more advanced math, and might be fatal if you try it before you're ready. So I'll just quote the result:

$$PE_E = \int \frac{\varepsilon_0}{2} E^2 \, dV \qquad \text{Still the energy stored in the electric field}$$

E here is the electric field strength, and the dV here stands for volume -- this is an integral over all space. The integrand is called the energy density (as it has units of J/m^3), and is symbolized:

$$u = \frac{\varepsilon_0}{2} E^2 \qquad \text{energy density}$$

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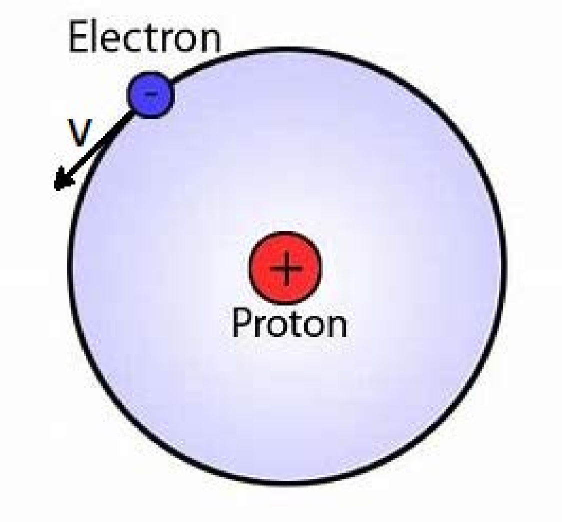
Ah, the hydrogen atom – every physicist's favorite, because it's the only one we can calculate. So remember how our electric orbits the proton with a radius roughly equal to *a* = 53pm. Let's calculate the total mechanical energy in this atom.



$$E_{mech.} = KE + PE$$

$$= \frac{1}{2}m_e v_e^2 + \frac{kq_1q_2}{r_{12}}$$

$$= \frac{1}{2}m_e v_e^2 - \frac{ke^2}{r}$$

to get velocity we said, $\sum F_c = m_e \frac{v_e^2}{r} \longrightarrow \frac{ke^2}{r^2} = m_e \frac{v_e^2}{r}$

we could directly solve for v, but being fancy, and recognizing this implies … $\frac{ke^2}{r} = m_e v_e^2$

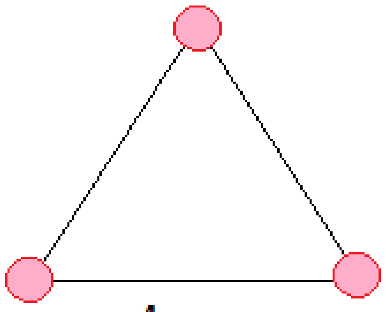
$$= \frac{1}{2}\frac{ke^2}{r} - \frac{ke^2}{r}$$

This *is* the actual energy of a hydrogen atom in its ground state.

$$= -\frac{1}{2}\frac{ke^2}{r} = -\frac{1}{2}\frac{(9\times10^9)(1.6\times10^{-19})^2}{(53\times10^{-12})} = -2.18\times10^{-18}\,\mathrm{J}$$

A.5 Electric Potential Energy

Three identical 5nC charges are fixed to the vertices of a 4cm sided equilateral triangle. What is their total potential energy, and if they're released, how fast will they be going when they're all very far apart? Can take their masses to be m = 5×10^{-20}kg.



So potential energy of the field is given by:

$$PE_E = \frac{kq_1q_2}{r_{12}} + \frac{kq_1q_3}{r_{13}} + \frac{kq_2q_3}{r_{23}}$$

$$= \frac{3kq^2}{r} = \frac{3(9\times10^9)(5\times10^{-9})^2}{0.04} = 1.7\times10^{-5}\,\mathrm{J}$$

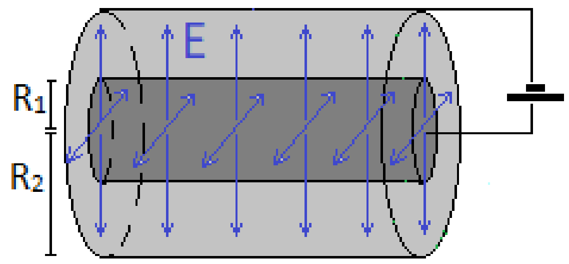
And if we released them all simultaneously, all of this potential energy would ultimately be converted to kinetic energy. So we'd have:

$$KE = PE_E$$

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2 = PE$$

$$\frac{3}{2}mv^2 = PE \longrightarrow v = \sqrt{\frac{2PE}{3m}} = \sqrt{\frac{2(1.7\times10^{-5})}{3(5\times10^{-20})}} = 1.5\times10^7\,\mathrm{m/s}$$

A.5 Electric Potential Energy



Our friend Mr. Geiger Counter is back. His inner radius is R_1 = 5cm, and outer radius R_2 = 10cm. Moreover, he constantly feeling like C = 12C/m^5 today. And….just after getting out of bed (when you're tallest) he's got a length ℓ = 30cm. Don't forget his electric field is given by that stuff on the right. How much energy does he have to start his day?

$$E = \begin{cases} \dfrac{C}{4\varepsilon_0} r^3 & r < R_1 \\ \dfrac{C}{4\varepsilon_0}\dfrac{R_1^4}{r} & R_1 < r < R_2 \\ 0 & R_2 < r \end{cases}$$

$$PE_E = \int u dV = \int \frac{\varepsilon_0}{2} E^2 dV \quad = \int_0^{R_1} \frac{\varepsilon_0}{2} E^2 dV_{cylinder} + \int_{R_1}^{R_2} \frac{\varepsilon_0}{2} E^2 dV_{cylinder} + \int_{R_2}^{\infty} \frac{\varepsilon_0}{2} E^2 dV_{cylinder}$$

$$= \int_0^{R_1} \frac{\varepsilon_0}{2}\left(\frac{C}{4\varepsilon_0} r^3\right)^2 \cdot 2\pi r l dr + \int_{R_1}^{R_2} \frac{\varepsilon_0}{2}\left(\frac{C}{4\varepsilon_0}\frac{R_1^4}{r}\right)^2 \cdot 2\pi r l dr + \int_{R_2}^{\infty} \frac{\varepsilon_0}{2}(0)^2 \cdot 2\pi r l dr$$

$$= \frac{\varepsilon_0}{2}\cdot\left(\frac{C}{4\varepsilon_0}\right)^2 \cdot 2\pi l \cdot \left[\int_0^{R_1} r^7 dr + \int_{R_1}^{R_2} \frac{R_1^8}{r} dr\right]$$

$$= \frac{\varepsilon_0}{2}\cdot\left(\frac{C}{4\varepsilon_0}\right)^2 \cdot 2\pi l \cdot \left[\frac{R_1^8}{8} + R_1^8 \ln\left(\frac{R_2}{R_1}\right)\right] \quad = 31\text{J}$$